THE DISTRIBUTIVE PROPERTY

We're becoming quite skillful at solving equations. But like a mutating virus, new varieties of equations are always confronting us. For example, how would we solve the equation

$$8(n - 1) = -(3n + 7)\ ?$$

The Distributive Property is the key to unlocking the variable when it is locked in parentheses.

This equation is distinctive for two reasons: the parentheses, and the fact that the variable *n* is locked up in two different sets of parentheses – on different sides of the equation, no less. If we could just open up the parentheses, we'd have access to the variable and thus be able to isolate it. In this chapter, we will learn how to unlock the parentheses so that we can solve the above equation.

❑ *STRETCHING AND SQUISHING*

To begin, recall that the single term 4w can be stretched out to four terms like this:

$$4w = w + w + w + w \qquad \text{[just as } 4 \times 5 = 5 + 5 + 5 + 5\text{]}$$

In the following examples we "stretch" out the multiplication and then "squish" the result by *combining like terms*:

$$2 \cdot (x + y) = x + y + x + y = \underline{2x + 2y}$$

$$3 \times (a - b) = a - b + a - b + a - b = \underline{3a - 3b}$$

$$4(w + 5) = w + 5 + w + 5 + w + 5 + w + 5 = \underline{4w + 20}$$

$3(2 - x) = 2 - x + 2 - x + 2 - x = \underline{6 - 3x}$, or $-3x + 6$

$2(m + n - q) = m + n - q + m + n - q = \underline{2m + 2n - 2q}$

$3(cd) = cd + cd + cd = \underline{3cd}$

$2(5n) = 5n + 5n = \underline{10n}$

$3(x \cdot 3) = 3(3x) = 3x + 3x + 3x = \underline{9x}$

$4(ab + 7) = ab + 7 + ab + 7 + ab + 7 + ab + 7 = \underline{4ab + 28}$

Homework

1. Use the method of the examples above (stretch-and-squish) to simplify each expression, i.e., remove parentheses:

a. $2 \cdot (x - 3)$	b. $3 \times (a + 5)$	c. $4(w + z)$
d. $2(AQ)$	e. $3(4 \cdot a)$	f. $4(uz)$
g. $2(x + 7)$	h. $3(1 - x)$	i. $4(t + 10)$
j. $2(a + b - c)$	k. $3(xy - 12)$	l. $5(x + 7)$

❑ *The Distributive Property*

You're in charge of food for the XYZ Widget, Inc. company picnic. You decide to serve box lunches, where each box contains

2 sandwiches
plus 5 cookies
plus 1 apple

You need to make <u>100</u> box lunches. That will require a total of

200 sandwiches
plus 500 cookies
plus 100 apples

How were these totals obtained? It should be pretty clear that we multiplied 100 by the number of each food item in one box lunch. That is, the quantity 100 was multiplied by 2 sandwiches, then by 5 cookies, and then by 1 apple:

100 (box lunches) = **100** (2 sandwiches + 5 cookies + 1 apple)

= 200 sandwiches + 500 cookies + 100 apples

In a sense, 100 has been "distributed" to all three food items in the box lunch. In fact, if we let

s = sandwiches
c = cookies
a = apples

then we can write

$$100\underbrace{(2s+5c+a)}_{\text{One box lunch}} = 200s+500c+100a$$

This statement is an example of the Distributive Property, which combines multiplication and addition into one rule, and is probably **THE MOST IMPORTANT STATEMENT IN ALL OF ALGEBRA.**

We can summarize the Distributive Property in the following way:

$$A(B+C) = AB+AC$$

<u>Note #1:</u> If we have the expression $A(B-C)$, we can distribute as follows: $A(B-C) = A\big(B+(-C)\big) = AB + A(-C) = AB - AC$. In other words, just as you might expect,

$$A(B-C) = AB-AC$$

Note #2: There can be more than two terms in the parentheses, and we can still distribute; for example,

$$\boldsymbol{a}(b + c - d) = \boldsymbol{a}b + \boldsymbol{a}c - \boldsymbol{a}d$$

Note #3: First recall the concept of *terms*: "abc" is one term, while "$a + b + c$" is three terms. To apply the distributive property, there must be <u>at least two terms</u> in the parentheses. We <u>can</u> apply the distributive property to the expression $x(y + z)$, since there are two terms in the parentheses (and the answer is $xy + xz$). But we <u>cannot</u> apply the distributive property to the expression $x(y \cdot z)$ because there is only one term in the parentheses. [By the way, $x(y \cdot z)$ simplifies to xyz.] Basically, if there's no addition or subtraction in the parentheses, there can be no addition or subtraction in the answer.

Summary:

$3(4 + x)$ — Distribute? <u>Yes</u>: $3(4 + x) = 12 + 3x$

$3(4 \cdot x)$ — Distribute? <u>No</u>: $3(4 \cdot x) = 12x$

EXAMPLE 1: **Apply the distributive property to each expression:**

A. $\mathbf{7}(5 + 3)$

$$= \mathbf{7} \cdot 5 + \mathbf{7} \cdot 3 = 35 + 21 = \mathbf{56}$$

This problem could also be worked by doing the addition in the parentheses first, via the Order of Operations:

$7(5 + 3) = 7(8) = 56$ — this problem should convince you that the distributive property really works.

B. $2(x + 9)$

$= 2x + 2(9) = \mathbf{2x + 18}$

C. $4(3n + 5)$

$= 4(3n) + 4(5) = \mathbf{12n + 20}$

D. $7(5a - 3)$

$= 7(5a) - 7(3) = \mathbf{35a - 21}$

E. $7(a \cdot b)$ = Whoa! Hold it right there.

Are there at least two terms in the parentheses? NO, so we may not apply the distributive property. [See the Caution sign above] All we can say is that $7(a \cdot b) = \mathbf{7ab}$.

F. $-3(7x + 8)$

$= -3(7x) - 3(8) = \mathbf{-21x - 24}$

G. $-10(7 - 6A)$

$= -10(7) - 10(-6A) = -70 + 60A = \mathbf{60A - 70}$

H. $(19u - 20v)$

$= 1(19u - 20v) = 1(19u) - 1(20v) = \mathbf{19u - 20v}$

In other words, when a quantity sits in a pair of parentheses, with nothing out in front, you may simply remove the parentheses.

I. $-(7x + 8)$

$= -1(7x + 8) = -1(7x) - 1(8) = \mathbf{-7x - 8}$

Notice that we can skip the use of the -1 and just distribute the minus sign in front of the parentheses to each term inside the parentheses.

J. $-(-3n - 12)$

$= -1(-3n - 12) = -1(-3n) - 1(-12) = \mathbf{3n + 12}$

K. $-(13a - 99)$

$= -1(13a - 99) = -1(13a) - 1(-99) = \mathbf{-13a + 99}$

L. $3(2x + 3y - 10z)$

$= 3(2x) + 3(3y) - 3(10z) = \mathbf{6x + 9y - 30z}$

Homework

2. Use the distributive property (if appropriate) to simplify each expression:

a. $2(x + 5)$	b. $3(y - 1)$	c. $5(2 + n)$
d. $6(2 \cdot m)$	e. $10(4 \times z)$	f. $7(\mathrm{T} - 7)$
g. $8(2x + 3)$	h. $12(a + b)$	i. $20(a \cdot b)$

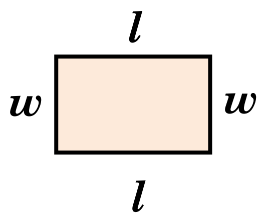
3. Use the distributive property (if appropriate) to simplify each expression:

a. $3(7x - 9)$	b. $3(a + 10)$	c. $-(2q + 8)$
d. $-2(-3n + 1)$	e. $-4(2 - y)$	f. $-(-x - 5)$
g. $7(5 - 3w)$	h. $-(3x - 9)$	i. $-5(3T - 8)$
j. $-3(-3 - 3n)$	k. $4(4 + 4u)$	l. $-(2v - 7)$
m. $20(x - 3)$	n. $-10(10 - 3n)$	o. $-(18 - 3a)$
p. $5(x \cdot 3)$	q. $-7(w \times z)$	r. $2(x - 3y + 7)$
s. $3(a \times b)$	t. $-2(x \cdot y \cdot z)$	u. $-(3x - 2y - 4z)$

❑ *CONNECTIONS*

1. The Distributive Property and the Perimeter of a Rectangle

Here's a more concrete way to approach the distributive property. Recall the formula for the perimeter of a rectangle:



$$P = 2l + 2w$$

In other words, we agreed that the sum of all four sides of the rectangle was simply 2 lengths plus 2 widths. But shouldn't we get the same result if we add one length and one width and then double that result? That is, couldn't the perimeter formula have been written $P = 2(l + w)$? Yes, it could. This leads us to conclude that

$$2(l + w) = 2l + 2w$$

which is simply another example of the distributive property at work.

2. The Distributive Property and Combining Like Terms

Here's another connection, this time between the distributive property from this chapter and the combining of like terms from another chapter.

Suppose we are asked to simplify the following expression by combining like terms:

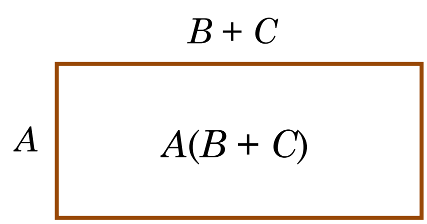
$$10x + 7x$$

We could use the common-sense notion that 10 of something plus 7 of the same somethings should result in 17 of those somethings: $10x + 7x = \underline{17x}$. But we can use the distributive property (in reverse) to get the same result:

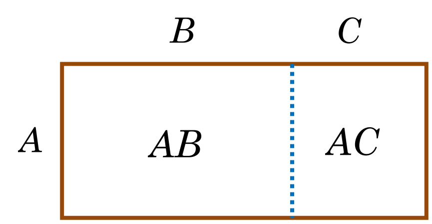
$$10x + 7x \;=\; x(10 + 7) \;=\; x(17) \;=\; \underline{17x}$$

3. The Distributive Property and the Area of a Rectangle

We conclude this section will still another view of the distributive property:



Rectangle #1



Rectangle #2

The area of Rectangle #1 is its width times its length:

$$A(B + C)$$

The area of Rectangle #2 can be thought of as the sum of the areas of the two smaller rectangles contained within it:

$$AB + AC$$

However, Rectangle #1 and Rectangle #2 are the same size (since they have the same dimensions), so their <u>areas must be the same</u>. And thus we see that

$$A(B + C) \;=\; AB + AC,$$

which is none other than the distributive property again.

❑ *DISTRIBUTING & COMBINING LIKE TERMS IN THE SAME PROBLEM*

<u>EXAMPLE 2:</u> **Simplify each expression completely by applying the distributive property and then combining like terms:**

A. $2x + 3(7x - 9) = 2x + 21x - 27 = \mathbf{23x - 27}$

B. $3a - 7(a - 1) = 3a - 7a + 7 = \mathbf{-4a + 7}$

C. $5b - 8 + 2(3 - b) = 5b - 8 + 6 - 2b = \mathbf{3b - 2}$

D. $(7w + 9) - (3w + 6) = 7w + 9 - 3w - 6 = \mathbf{4w + 3}$

E. $-3(x - 1) + 3(2x + 4) = -3x + 3 + 6x + 12 = \mathbf{3x + 15}$

F. $4(-3n + 7) - 4(5n - 9) = -12n + 28 - 20n + 36 = \mathbf{-32n + 64}$

Homework

4. Apply the distributive property to remove the parentheses and then simplify your answer by combining like terms:

a. $6 + 3(x - 1)$

b. $3(2m + 3) + 5(1 - m)$

c. $2(x - 3) - (2x - 3)$

d. $-(3n - 1) - (1 - 3n)$

e. $-4(4t + 1) - 3(3t - 2)$

f. $3(a - b) + 4(b - a)$

g. $3(2y + 1) - 2(y - 1)$

h. $-(1 - x) - (x - 1)$

i. $3(4x - 1) - (12x - 1)$

j. $5 - 4(x - 3)$

k. $10 + 2(3x - 1) - x - 1$

l. $2(x + 3) - 2(x - 5)$

Review Problems

5. Simplify each expression:

a. $-2(3x - 4y)$ b. $7(7a + 8c - 1)$ c. $-4(3 - 4u + h - k)$

d. $-(4x - y)$ e. $-(9 + 12w)$ f. $-(-e - t - p)$

g. $8(9 \cdot y)$ h. $-2(-8 \times z)$ i. $-(ab)$

j. $17(xy + z)$ k. $-10(a \cdot b \cdot c)$ l. $4(3uw)$

m. $-2(-2x - 2y)$ n. $25(2 \times u \times w)$ o. $-2(g \cdot h)$

p. $7(3x \cdot 3)$ q. $7(3x - 3)$ r. $7(3x + 3)$

s. $-5(4n + 6)$ t. $-5(4n \cdot 6)$ u. $-5(4n \times 6)$

6. Simplify each expression:

a. $3(x + 4) + 5(2x - 1)$ b. $-2(n - 1) - 4(1 - n)$

c. $3(2y + 1) - 2(y - 1)$ d. $-(1 - x) - (x - 1)$

e. $3(4a + 1) - (10a - 2)$ f. $5 - 3(w - 4)$

g. $10 + 2(3h - 1) - 2h - 1$ h. $2(k + 4) - 2(k - 5)$

Solutions

1. a. $2 \cdot (x - 3) = x - 3 + x - 3 = 2x - 6$ b. $3a + 15$

c. $4w + 4z$ d. $2(AQ) = AQ + AQ = 2AQ$

e. $12a$ f. $4uz$ g. $2x + 14$

h. $3 - 3x$ i. $4t + 40$ j. $2a + 2b - 2c$

k. $3xy - 36$ l. $5x + 35$

2. a. $2x + 10$ b. $3y - 3$ c. $10 + 5n$, or $5n + 10$

d. $12m$ e. $40z$ f. $7T - 49$

g. $16x + 24$ h. $12a + 12b$ i. $20ab$

3. a. $21x - 27$ b. $3a + 30$ c. $-2q - 8$ d. $6n - 2$

e. $-8 + 4y$ f. $x + 5$ g. $35 - 21w$ h. $-3x + 9$

i. $-15T + 40$ j. $9 + 9n$ k. $16 + 16u$ l. $-2v + 7$

m. $20x - 60$ n. $-100 + 30n$, or $30n - 100$

o. $-18 + 3a$, or $3a - 18$ p. $15x$ q. $-7wz$

r. $2x - 6y + 14$ s. $3ab$ t. $-2xyz$

u. $-3x + 2y + 4z$, or $2y + 4z - 3x$

4. a. $6 + 3x - 3 = 3x + 3$ b. $6m + 9 + 5 - 5m = m + 14$

c. $2x - 6 - 2x + 3 = -3$ d. $-3n + 1 - 1 + 3n = 0$

e. $-25t + 2$ f. $b - a$, or $-a + b$

g. $4y + 5$ h. 0

i. -2 j. $-4x + 17$, or $17 - 4x$

k. $5x + 7$ l. 16

5. a. $-6x + 8y$ b. $49a + 56c - 7$ c. $-12 + 16u - 4h + 4k$

d. $-4x + y$ e. $-9 - 12w$ f. $e + t + p$

g. $72y$ h. $16z$ i. $-ab$

j. $17xy + 17z$ k. $-10abc$ l. $12uw$

m. $4x + 4y$ n. $50uw$ o. $-2gh$

p. $63x$ q. $21x - 21$ r. $21x + 21$

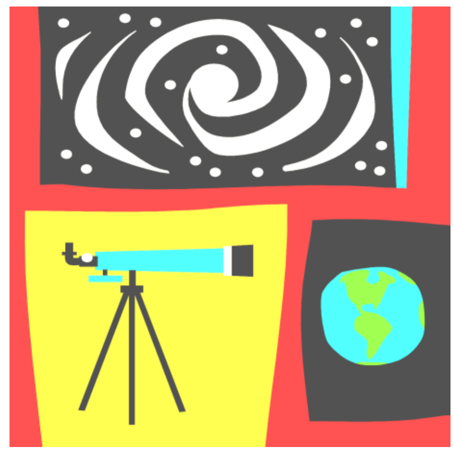
s. $-20n - 30$ t. $-120n$ u. $-120n$

6. a. $3(x + 4) + 5(2x - 1) = 3x + 12 + 10x - 5 = \mathbf{13x + 7}$

b. $-2(n-1)-4(1-n) = -2n+2-4+4n = \mathbf{2n-2}$

c. $4y+5$ d. 0 e. $2a+5$ f. $-3w+17$ g. $4h+7$ h. 18

"For most of human history we have searched for our place in the cosmos. Who are we? What are we? We find that we inhabit an insignificant planet of a hum-drum star lost in a galaxy tucked away in some forgotten corner of a universe in which there are far more galaxies than people. We make our world significant by the courage of our questions, and by the depth of our answers."



—Carl Sagan (1934–1996)